ARC SPACES OF cA_1 SINGULARITIES

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For a k-variety X let $X_\infty := \operatorname{Mor}(\operatorname{Spec} k[[t]], X)$ denote the *space of arcs*. Note that X_∞ is a (non-noetherian) scheme. Assume for simplicity that $(x \in X)$ is an isolated singularity and let $X_\infty(x) \subset X_\infty$ denote those arcs that pass through x.

A preprint of Nash, written in 1968 but only published later as [Nas95], describes an injection (now called the *Nash map*) from the irreducible components of $X_\infty(x)$ to the set of so called *essential divisors*. These are the divisors whose center on X equals $\{x\}$ and that appear on every resolution of X. The *Nash problem* asks if this map is also surjective or not. This was shown to fail in dimensions ≥ 4 [IK03] but it is true in dimension 2 [FdBP11]. The recent paper [dF12] proves that the Nash map need not be surjective in dimension 3.

The aim of this note – which grew out of my attempt to understand what is behind some computations of [dF12] – is to give simpler examples in dimension 3. In essence, the simplest possible singularities already work. The following is obtained by combining Propositions 2 and 3.

Example 1. The Nash map is not surjective for $X_m := (x^2+y^2+z^2+t^m = 0) \subset \mathbb{C}^4$ for all odd $m \geq 5$.

On the one hand, this can be interpreted to mean that the Nash problem hopelessly fails in dimension 3. On the other hand, the proof leads to a reformulation of the Nash problem and to an approach that seems feasible in dimension 3 (16).

From now on we work over an algebraically closed field k whose characteristic is different form 2. In order to enumerate both sides of the Nash map, we start by exhibiting some singularities $(0 \in X)$ for which $X_\infty(0)$ is irreducible.

Proposition 2. *Let $(0 \in X)$ be an isolated cA_1-type singularity of dimension ≥ 3. Then $X_\infty(0)$, the space of arcs through 0, is irreducible.*

In some coordinates write a hypersurface singularity $(0 \in X)$ as $\bigl(f(x_1, \dots, x_{n+1}) = 0\bigr)$. Let f_2 denote the quadratic part of f. Then $(f_2 = 0)$ is the tangent cone of X at the origin and cA_1-type means that $\operatorname{rank} f_2 \geq 3$.

Note that by adding 3 squares in new variables we get a map from hypersurface singularities in dimension n (modulo isomorphism) to cA_1-type hypersurface singularities in dimension $n+3$ (modulo isomorphism). This map is one-to-one and onto; see [AGZV85, Sec.11.1]. Thus cA_1-type singularities are quite complicated in large dimensions.

In dimension 3, the only cA_1-type singularities are $X_m := (x^2+y^2+z^2+t^m = 0)$ for $m \geq 2$. Already [Nas95, p.37] proved that they have at most 2 essential divisors. Then we use the method of [dF12, 4.1] to determine the precise count.

Proposition 3. *Set $X_m := (x^2 + y^2 + z^2 + t^m = 0) \subset \mathbb{C}^4$.*

(1) *If $m \geq 5$ is odd, there are 2 essential divisors.*
(2) *If $m \geq 2$ is even or $m = 3$, there is 1 essential divisor.*

Even in dimension 3, it seems surprisingly difficult to determine the set of essential divisors. If X has terminal hypersurface singularities then all divisors E with $a(E, X) = 1$ are essential and the method of [dF12, 4.1] gives a good way to study the $a(E, X) = 2$ cases. By contrast, the computations in [Nas95, p.37] (see also Lemma (12)) seem to be just lucky. The papers [Hay05a, Hay05b] contain important related results.

Proof of Proposition 2.

In this section $(0 \in X)$ denotes a cA_1-type singularity and $\phi : \operatorname{Spec} k[[t]] \to X$ an arc whose image is not contained in $\operatorname{Sing} X$.

Let $\pi : B_0X \to X$ denote the blow-up of the origin and $E \subset B_0X$ the exceptional divisor. Then ϕ lifts to an arc $\tilde{\phi} : \operatorname{Spec} k[[t]] \to B_0X$. Let $p \in E$ denote the image of the closed point.

The arc $\phi : \operatorname{Spec} k[[t]] \to X$ is called *smooth* if ϕ is unramified and *general* if p is a smooth point of E. Equivalently, if the tangent cone of X is smooth along the tangent direction of ϕ.

Since $\operatorname{rank} f_2 \geq 3$, E is irreducible and its smooth locus is connected. Thus general smooth arcs form an irreducible family. It is therefore enough to prove the following.

Proposition 4. *Let $(0 \in X)$ be an isolated cA_1-type singularity of dimension ≥ 3. Then the general smooth arcs are dense in $X_\infty(0)$.*

By [FdBP12] this implies that every arc is a limit of general smooth arcs. That is, given any $\phi : k[[t]] \to X$ there is an extension of it to $\Phi : k[[t, s]] \to X$ such that the induced $\Phi_s : k((s))[[t]] \to X$ is a general smooth arc.

The key steps are the next two lemmas.

Lemma 5. *Let $\phi : \operatorname{Spec} k[[t]] \to X$ be a general smooth arc. Then, in suitable coordinates we can write the equations as*

$$X := \bigl(x_1x_2 + x_3^2 + F(x_4, \dots, x_{n+1}) = 0\bigr) \quad \textit{and} \quad \phi^*x_1 = t, \phi^*x_i = 0 : i > 1.$$

Proof. Since ϕ is a smooth arc, $\phi^*x_1 = t$ and $\phi^*x_i = 0 : i > 1$ are easy to arrange. Then we apply the Morse lemma with parameters as in [AGZV85, Sec.6.2] to get the usual normal form as above.

Since we need only a formal coordinate change, this can also be done as a step-by-step method as follows.

Assume that we already have the equation of X in the form

$$x_1x_2 + x_3^2 + F_m(x_4, \dots, x_{n+1}) + H_m(x_1, \dots, x_{n+1}) = 0$$

where $H_m \in (x_1, x_2, x_3) \cdot (x_1, \dots, x_{n+1})^m$. Since the x_1-axis is contained in X, no term in H_m is a pure x_1-power. Thus we can write the degree $m+1$ part of H_m as

$$x_1\phi_2 + x_2\phi_1 + 2x_3\phi_3 \quad \text{where} \quad \phi_i \in (x_1, \dots, x_{n+1})^m$$

and the ϕ_i vanish along the x_1-axis. Thus in the new coordinates

$$(x_1', \dots, x_{n+1}') = (x_1 - \phi_1, x_2 - \phi_2, x_3 - \phi_3, x_4, \dots, x_{n+1})$$

the equation of X becomes

$$x_1'x_2' + {x_3'}^2 + F_{m+1}(x_4', \dots, x_{n+1}') + H_{m+1}(x_1', \dots, x_{n+1}') = 0$$

where $H_{m+1} \in (x_1', x_2', x_3') \cdot (x_1', \dots, x_{n+1}')^{m+1}$. $\square$

Lemma 6. *Let ϕ be an arc that becomes smooth and general after one blow-up. Then ϕ is a limit of general smooth arcs.*

Proof. Let $E \subset B_0X$ be the exceptional divisor and $p \in B_0X$ the center of $\tilde{\phi}$. Apply (5) and choose coordinates x'_i such that $\operatorname{im}\tilde{\phi}$ is the x'_1-axis and B_0X is locally given by $x'_1x'_2 + {x'_3}^2 + F(x'_4, \dots, x'_{n+1}) = 0$.

Note that E is a quadric of rank ≥ 3, thus its singular set has codimension ≥ 2. Since $E \cap (x'_2 = 0)$ has dimension $\geq \dim E - 1$, we can choose a parametrized curve in it given by

$$\tau^*(x'_1, \dots, x'_{n+1}) = (\tau_1(s), 0, \tau_3(s), \dots, \tau_{n+1}(s))$$

such that $\tau(0) = p$ and E is smooth elsewhere along the image of τ. Then

$$\Phi^*(x'_1, \dots, x'_{n+1}) := (t + \tau_1(s), 0, \tau_3(s), \dots, \tau_{n+1}(s))$$

is a deformation of $\tilde{\phi}$ to an arc through a smooth point of E. We can then make the arc transversal to E hence its image in X is a general smooth arc. □

7 (Proof of (4))**.** Given any arc ϕ we want to write it as a limit of general smooth arcs. The proof is by induction on the number of blow-ups needed to make the blow-up of X smooth along the lift of ϕ.

For 1 blow-up we can perturb $\tilde{\phi}$ freely to pass through a smooth point of E.

In general, we get $\tilde{\phi}$ mapping to B_0X which can be resolved by fewer blow-ups. Thus, by induction, $\tilde{\phi}$ is a limit of general smooth arcs $\tilde{\phi}_s$, hence ϕ is a limit of arcs ϕ_s that become smooth after 1 blow-up. By (6), each ϕ_s is a limit of general smooth arcs. □

Proof of Proposition 3.

8 (Resolving X_m)**.** Blow up the origin to get $\pi_1 : X_{m,1} := B_0X_m \to X_m$. The exceptional divisor is the singular quadric $E_1 \cong (xy + z^2 = 0) \subset \mathbb{P}^3(x, y, z, t)$. B_0X_m has one singular point, visible in the chart

$$(x_1, y_1, z_1, t) := \big(x/t, y/t, z/t, t\big)$$

where the local equation is $x_1y_1 + z_1^2 - t^{m-2} = 0$. We can thus blow up the origin again and continue. After $r := \lfloor \frac{m}{2} \rfloor$ steps we have a resolution

$$\Pi_r : X_{m,r} \to X_{m,r-1} \to \cdots \to X_{m,1} \to X_m.$$

We get r exceptional divisors $E_r, \dots, E_1$. For $1 \leq c \leq r$ the divisor E_c first appears on $X_{m,c}$. At the unique singular point one can write the local equation as

$$X_{m,c} = \big(x_cy_c + z_c^2 - t^{m-2c} = 0\big) \quad \text{and} \quad E_c = (t = 0).$$

where

$$(x_c, y_c, z_c, t) := \big(x/t^c, y/t^c, z/t^c, t\big)$$

We thus need to decide which of the divisors $E_1, \dots, E_{\lfloor \frac{m}{2} \rfloor}$ are essential. It is easy to see that E_1 is essential and a lucky computation (12) shows that $E_3, \dots, E_{\lfloor \frac{m}{2} \rfloor}$ are not. (This is actually not needed in order to establish Example 1.) The hardest is to decide what happens with E_2.

Lemma 9. *E_1 appears on every resolution of X_m whose exceptional set is a divisor.*

More generally, let $p : Y \dashrightarrow X_m$ be any (not necessarily proper) birational map from a smooth variety Y such that $\operatorname{center}_Y E_1 \subset Y$ *is not empty. Then* $\operatorname{center}_Y E_1$ *is an irreducible component of the exceptional set* $\operatorname{Ex}(p)$.

Proof. Set $W_1 := \operatorname{center}_Y E_1 \subset Y$. Let $F_i \subset Y$ be the exceptional divisors and note that

$$a(E_1, X_m) \geq \big(\operatorname{codim}_Y W_1 - 1\big) + \textstyle\sum_i \operatorname{mult}_{W_1} F_i \cdot a(F_i, X_m). \tag{9.1}$$

Note that X_m is terminal, $a(E_1, X_m) = 1$ and $a(F_i, X_m) \geq 1$ for every i.

If W_1 is not an irreducible component of $\operatorname{Ex}(p)$ then $W_1 \subset F_i$ form some i and then both terms on the right hand side of (9.1) are positive, a contradiction. □

Lemma 10. *If $m \in \{2,3\}$ then B_0X is smooth, hence the only essential divisor is E_1.* □

Lemma 11. *If m is even then there is a divisorial resolution whose sole exceptional divisor is birational to E_1. Thus the only essential divisor is E_1.*

Proof. The $m = 2$ case is in (10), hence we assume that $m \geq 4$ and set $a = (m-2)/2$.

Then B_0X_m has one singular point with local equation $x_1y_1 + z_1^2 - t_1^{m-2} = 0$.

Next blow up $D := (x_1 = z_1 - t_1^a = 0)$ to get $\tau : B_DB_0X_m \to B_0X_m$. We see that $B_DB_0X_m$ is smooth and $\pi \circ \tau : B_DB_0X_m \to X_m$ has 1 smooth exceptional divisor which is birational to E_1. □

Lemma 12. [Nas95, p.37] *The divisors $E_3, \dots, E_r$ are not essential.*

Proof. If m is even, this follows from (11), but for the proof below the parity of m does not matter.

If $2b \geq a \geq 0$ and $m \geq a$ then $(u, v, w, t) \mapsto (ut, vt^{a+1}, wt^{b+1}, t) = (x, y, z, t)$ defines a birational map

$$g(a, b, m) : Z_{abm} := \big(uv + w^2t^{2b-a} - t^{m-2-a} = 0\big) \to X_m.$$

Note that $\operatorname{Ex}\big(g(a, b, m)\big) = (t = 0)$ is mapped to the origin and Z_{abm} is smooth along the v-axis, save at the origin.

If $1 \leq c \leq m/2$ then $(x_c, y_c, z_c, t) \mapsto (x_ct^c, y_ct^c, z_ct^c, t) = (x, y, z, t)$ defines a birational map

$$h(c, m) : X_{m,c} := \big(x_cy_c + z_c^2 - t^{m-2c} = 0\big) \to X_m.$$

By composing we get a birational map $g(a, b, m)^{-1} \circ h(c, m) : Y_c \dashrightarrow Z_{abm}$ given by

$$(x_c, y_c, z_c, t) \mapsto (x_ct^{c-1}, y_ct^{c-a-1}, z_ct^{c-b-1}, t) = (u, v, w, t)$$

which is a morphism if $c \geq a+1, b+1$. If $c = a+1$ and $c > b+1$ then we have

$$(x_c, y_c, z_c, t) \mapsto (x_ct^{c-1}, y_c, z_ct^{c-b-1}, t) = (u, v, w, t)$$

which maps E_c to the v-axis.

If $c \geq 3$ then by setting $a = c-1, b = c-2$ we get a birational morphism $p(c, m) := g(c, c-1, m)^{-1} \circ h(c, m)$ given by

$$(x_c, y_c, z_c, t) \mapsto (x_ct^c, y_c, z_ct, t) = (u, v, w, t).$$

Note that

$$p(c, m) : Y_c = \big(x_cy_c + z_c^2 - t^{m-2c} = 0\big) \to \big(uv + w^2t^{c-2} - t^{m-c} = 0\big) = Z_{c,c-1,m}$$

maps E_c onto the v-axis. Thus E_c is not essential for $c \geq 3$. □

Lemma 13. *If $m \geq 5$ is odd then E_2 is essential.*

The proof follows an argument in [dF12, 4.1]. Let $p : Y \to X_m$ be any resolution and set $W_2 := \operatorname{center}_Y E_2 \subset Y$. Since X_m is factorial (here we use that m is odd), $\operatorname{Ex}(p)$ has pure dimension 2.

Using that $a(E_2, X_m) = 2$, (9.1) implies that if W_2 is not a divisor then W_2 is a curve, there is a unique exceptional divisor $F \subset Y$ that contains W_2, F is a smooth at the general point of W_2 and $a(F, X_m) = 1$.

If $p(F)$ is a curve then W_2 is an irreducible component of $p^{-1}(0)$. The remaining case is when $p(F) = 0$, thus $F = E_1$.

Since t vanishes along E_2 with multiplicity 1, it also vanishes along W_2 with multiplicity 1. Since p^*x, p^*y, p^*z, t all vanish along E_1 so $p^*(x/t), p^*(y/t), p^*(z/t)$ are regular generically along W_2. Thus $\pi_1^{-1} \circ p : Y \dashrightarrow X_{m,1}$ is a morphism generically along W_2. Note that our E_2 is what we would call E_1 if we started with $X_{m,1}$. Applying (9) to $Y \dashrightarrow X_{m,1}$ we get a contradiction. $\square$

A revised version of the Nash problem.

In this section we use the proof of (6) to propose a revised version of the Nash problem.

Definition 14. Let X be a k-variety, $K \supset k$ a field extension and $\phi : \operatorname{Spec}_k K[[t]] \to X$ an arc such that $\operatorname{Supp} \phi^{-1}(\operatorname{Sing} X) = \{0\}$. A *sideways deformation* of ϕ is an extension of ϕ to a morphism $\Phi : \operatorname{Spec}_k K[[t, s]] \to X$ such that $\operatorname{Supp} \Phi^{-1}(\operatorname{Sing} X) = \{(0, 0)\}$.

We say that X is *arc-wise Nash-trivial* if every general arc in $X_\infty(\operatorname{Sing} X)$ has a sideways deformation.

The main steps of the counter example in [IK03] was to prove that the cone over a smooth cubic 3-fold is arc-wise Nash-trivial and then to argue that essentially any $(0 \in X)$ such that B_0X has a singularity that is arc-wise Nash-trivial is a counter example.

Definition 15. Let X be a k-variety. A divisor E over X is called *very essential* if the following holds. Let $p : Y \to X$ be a proper, birational morphism such that Y is $\mathbb{Q}$-factorial and has only arc-wise Nash-trivial singularities. Then $\operatorname{center}_Y E$ is an irreducible component of $p^{-1}(\operatorname{Sing} X)$.

It is easy to see that the Nash map is an injection from the irreducible components of $X_\infty(\operatorname{Sing} X)$ to the set of very essential divisors. One can hope that there are no other obstructions.

Problem 16 (Revised Nash problem)**.** Is the Nash map surjective onto the set of very essential divisors for normal 3-folds?

As a first step, one should consider the following.

Problem 17. In dimension 3, classify all arc-wise Nash-trivial singularities.

Hopefully they are all terminal and a complete enumeration is possible.

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